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FULLY DEVELOPED LAMINAR FREE CONVECTION FLOW BETWEEN TWO PARALLEL VERTICAL WALLS—I

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Abstract—Based on the non-linear variation of the density with temperature—(NDT) in the buoyancy term, the free convection flow and heat transfer between two parallel vertical walls is analysed. When the wall temperature ratio m or the free convection parameter K is moderately large the fluid velocities and temperatures for the NDT case exceed the corresponding values of the linear density variation (LDT) and this increase is more pronounced in the presence of the heat sources. As a special case we discuss the problem for the case of quadratic-density temperature-variation (QDT) when the heat sources are present.

NOMENCLATURE

β_0, β_1, constants in equation (3);
T, temperature;
y, dimensionless co-ordinate;
u, dimensionless velocity;
$\bar{u}$, dimensionless velocity in QDT case;
t, dimensionless temperature;
$\bar{t}$, dimensionless temperature in QDT case;
γ, NDT parameter;
Nu, Nusselt number;
$\overline{Nu}$, Nusselt number in QDT case.

Subscripts

s, hydrostatic condition;
w, wall condition;
w_0, condition at $y = 0$;
w_1, condition at $y = 1$.

1. INTRODUCTION

OSTRACH [1–2] has analysed the effect of the frictional heating and the heat sources in the fluid, on the fully developed laminar convection flow between two parallel vertical plates when the wall temperatures are either constant or vary linearly along the plate length. He has used the usual, linear, density–temperature variation namely

$$\Delta\rho = -\rho\beta(T-T_s). \qquad (1)$$

Goren [3] has obtained a similarity solution of the boundary layer equations of the free convection flow from a semi-infinite plate of uniform temperature to water at 4°C. In this study he has established the necessity of using in the buoyancy force term the quadratic-density temperature-variation namely

$$\Delta\rho = -\rho\gamma(T-T_s)^2. \qquad (2)$$

Sinha [4] has investigated the fully developed laminar convection flow between two parallel vertical plates, taking into account, after Goren [3], the quadratic density-temperature variation. Barrow and Seetharama Rao [5] have analysed the effect of β varying linearly with the temperature in the relation (1) on laminar free convection heat transfer from a constant temperature, vertical, flat plate. Brown [6] reconsidered the same problem to estimate the temperature-dependence of density (ignored in [5]) consequent upon the temperature-dependence of β. We can look at the variation of density with temperature from a different angle (see [7] and [8]) and write that

$$\rho(T) = \rho(T_s) + \left(\frac{\partial\rho}{\partial T}\right)_s (T-T_s) + \left(\frac{\partial^2\rho}{\partial T^2}\right)_s (T-T_s)^2 + \ldots.$$

By taking into account the terms up to $(T-T_s)^2$ we write the above relation as

$$\Delta\rho/\rho = -\beta_0(T-T_s) - \beta_1(T-T_s)^2 \qquad (3)$$

which may be named as the NDT variation. The relation (3) naturally accommodates the relations (1) and (2), and to some extent, takes care of the linear temperature-dependence of β, used by [5, 6]. Recently Gilpin [9] has used a density–temperature relation, which is similar to the foregoing relation (3) and which has been introduced by Vanier and Tien [10] with a view to predict the heat-transfer results in the case of water for temperatures between 0 and 20°C.

In this paper we intend to make use of the relation (3) and investigate the fully developed free convection flow and heat transfer between two long parallel vertical walls kept at constant temperatures (when the wall temperatures vary linearly along their length we re-investigate the same problem and report the results in Part II). We take into account the frictional heating and the heat sources in the flow. In this case we observe that the flow and heat transfer depend on a new dimensionless number $\gamma[=(\beta_1/\beta_0)\Delta T]$ in addition to the wall temperature ratio m and the free convection parameter K. Numerical calculations have been carried out for various values of the parameters and qualitatively significant contributions of the NDT parameter γ to the flow and heat-transfer characteristics have been pointed out.

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2. FORMULATION AND SOLUTION OF THE PROBLEM

Using the relation (3) in the buoyancy force term and following Ostrach [1] closely for non-dimensionalisation and other simplifications we reduce the basic equations that govern the flow and heat transfer of the title problem to

$$u''+t+\gamma t^2 = 0,\qquad t''+u'^2+\alpha K = 0 \tag{4}$$

where α is the heat source parameter and γ the NDT parameter. The pertinent boundary conditions in the non-dimensional form are

$$u(0) = 0 = u(1),\qquad t(0) = K = \frac{1}{m}t(1) \tag{5}$$

where $m = (T_{w_1}-T_s)/(T_{w_0}-T_s)$ = wall temperature ratio. Considering the effect of the nonlinear terms in equations (4) to the first order, the expressions for u and t (u_2, t_3: suffixes denote the order of iteration) are obtained subject to the boundary conditions (5) by using an iteration scheme as in [1]. The values of u_2 and t_3 and the Nusselt numbers at the walls, namely, $Nu_{0,1}$ are evaluated numerically for $K = 0.5$, 3, 10; $m = -1$, 1, 2; $\alpha = 0$, 10 and $\gamma = -0.2$, 0.05, 0.2. The results are presented in Figs. 1–6 and in Table 1. The deviations of the wall Nusselt numbers of NDT case from their LDT-counterparts are defined as

$$[Nu_{0,1}(\gamma)-Nu_{0,1}(\gamma=0)]/Nu_{0,1}(\gamma=0)$$

and their percentages abbreviated as Nu_{0DP} and Nu_{1DP} and calculated numerically for the above mentioned values of the various dimensionless parameters.

3. DISCUSSION

NDT case

At the outset let us analyse the flow and heat transfer characteristics as affected by the nonlinear variation of density with temperature (NDT) and make a comparative study with the linear variation of density with temperature (LDT). In Figs. 1–4 the velocity and temperature-distributions are shown for the two cases: (i) LDT case—when the NDT parameter γ is neglected, and (ii) NDT case—when it is taken into account. We observe that the velocities as well as the temperatures of the fluid are considerably increased with positive values of the NDT parameter γ and diminished when it takes negative values. This observation holds for all values of the free convection parameter K or the wall temperature ratio m or the heat source parameter α. Again this increase (decrease) in the velocities and temperature with positive (negative) γ is prominent when the free convection parameter or the wall temperature ratio or the heat source parameter takes higher values—a result qualitatively similar to that of Sinha [4]. In general the effect of the NDT-parameter on the fluid-velocity and temperature is more pronounced in the presence of heat sources and when the free convection parameter K is moderately large (see Fig. 3).

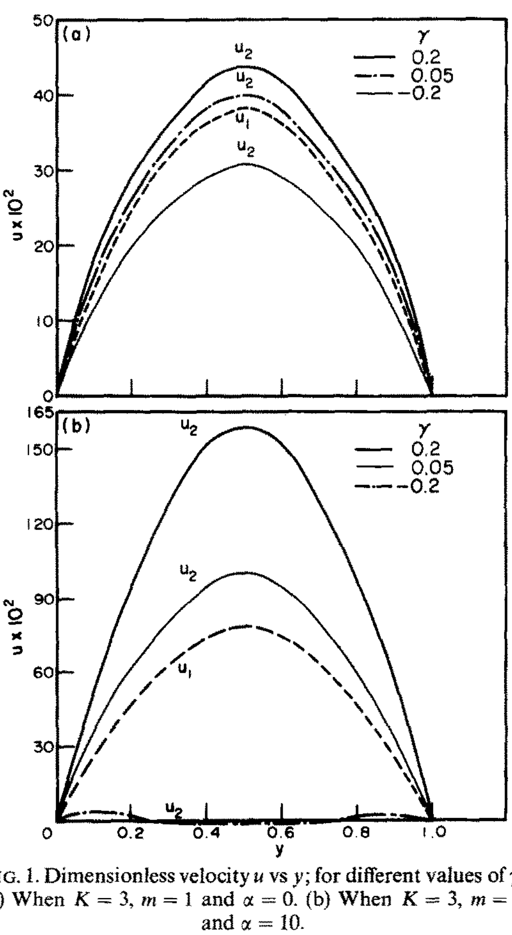

FIG. 1. Dimensionless velocity u vs y; for different values of γ. (a) When $K = 3$, $m = 1$ and $\alpha = 0$. (b) When $K = 3$, $m = 1$ and $\alpha = 10$.

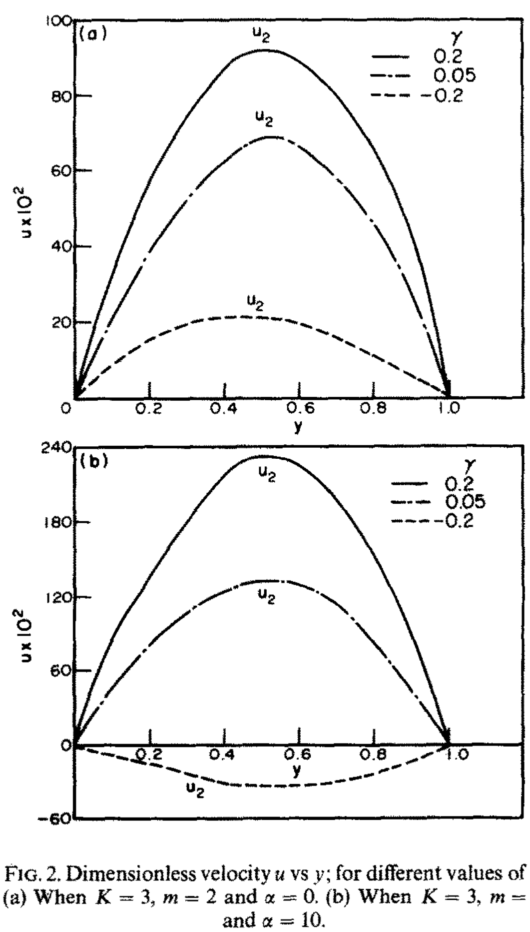

FIG. 2. Dimensionless velocity u vs y; for different values of γ. (a) When $K = 3$, $m = 2$ and $\alpha = 0$. (b) When $K = 3$, $m = 2$ and $\alpha = 10$.

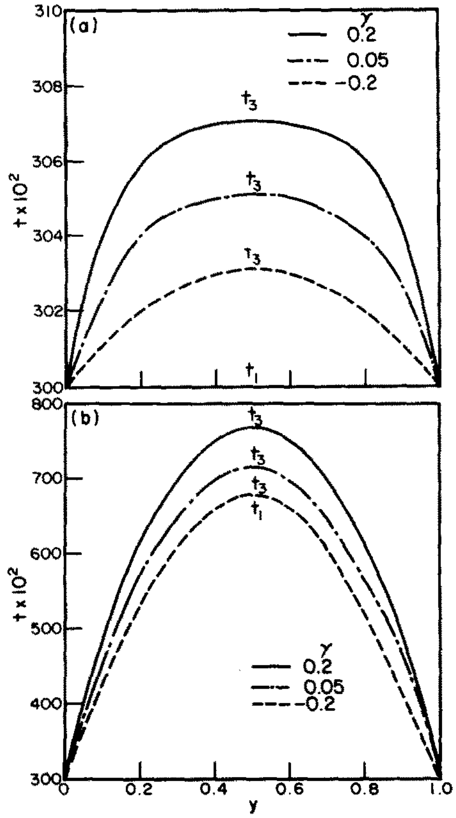

FIG. 3. Dimensionless temperature t vs y; for different values of γ. (a) When $K = 3$, $m = 1$ and $\alpha = 0$. (b) When $K = 3$, $m = 1$ and $\alpha = 10$.

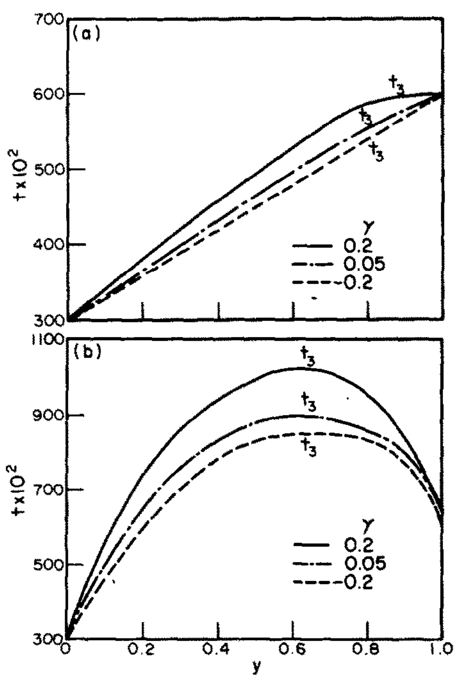

FIG. 4. Dimensionless temperature t vs y; for different values of γ. (a) When $K = 3$, $m = 2$ and $\alpha = 0$. (b) When $K = 3$, $m = 2$ and $\alpha = 10$.

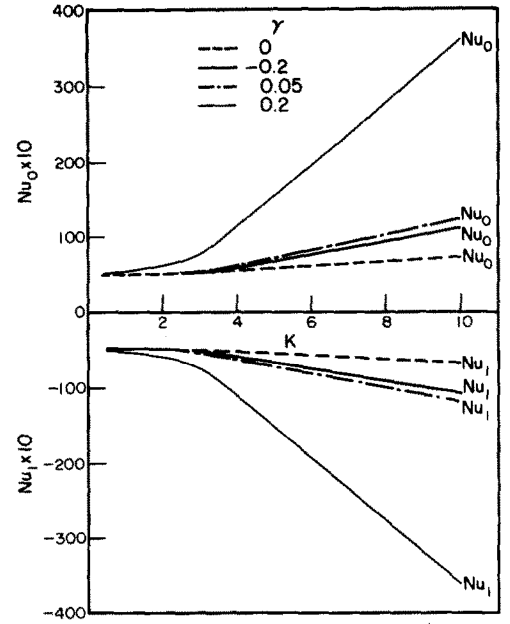

FIG. 5. Nusselt numbers at $y = 0$ and $y = 1$ vs K; for different values of γ, when $m = 1$ and $\alpha = 10$.

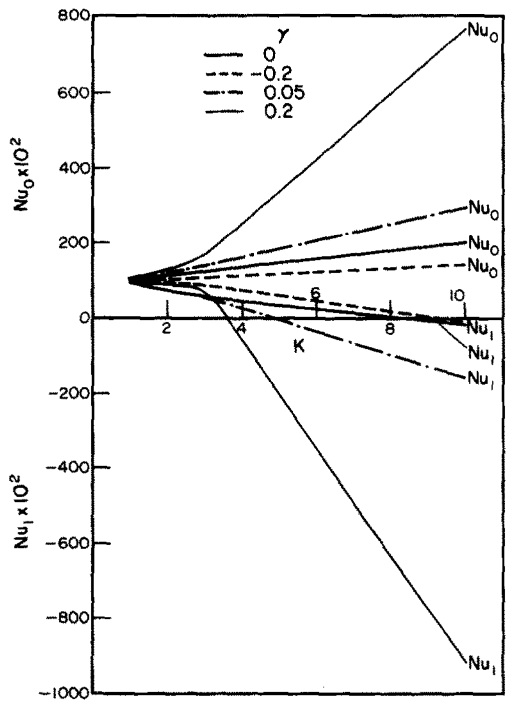

FIG. 6. Nusselt numbers at $y = 0$ and $y = 1$ vs K; for different values of γ, when $m = 2$ and $\alpha = 0$.

From Table 1 we notice that the Nusselt numbers at $y = 0$ are always positive and those at the other wall $y = 1$ are negative, except when $m = -1$ (but the difference of the Nusselt numbers at either wall is always positive (negative) for positive γ (negative γ) except when $m = -1$, that is, the values of the Nusselt numbers are definitely more pronounced in the presence of the NDT-parameter than when it is absent). Physically it means that the heat flows from the fluid into the wall and this heat flow is more rapid when the NDT-parameter is positive. In fact, in the presence of positive γ, the heat flow to the wall $y = 0$ can be as much as one and a half times that when γ is neglected (that at the other wall is nearly twice)—a

Table 1. Values of Nusselt numbers at the walls $y = 0$ and $y = 1$ and their percentage differences for NDT case

	γ	$\gamma = 0$			$\gamma = -0.2$			$\gamma = 0.2$		
		K								
m and α	Nu	0.5	3	10	0.5	3	10	0.5	3	10
	$Nu_0 = (1)$	0.9993	0.9958	0.9859	0.9995	0.9869	0.5006	0.9988	0.9946	0.6418
$m = -1$	$Nu_1 = (2)$	1.0007	1.0042	10.1369	1.0012	1.0055	13.6247	1.0005	1.0130	14.9428
and $\alpha = 0$	$Nu_{0DP} = (3)$				0.0207	−0.8960	−49.2220	−0.0509	−0.1225	−34.8997
	$Nu_{1DP} = (4)$				0.0510	0.1300	34.4060	−0.0210	0.8800	47.4090
	(1)	−1.5138	−1.5849	−1.8034	−1.5058	−1.5285	−1.5368	−1.5204	−1.6736	−1.7407
$m = -1$	(2)	3.5088	3.5548	36.9770	3.5053	3.5260	36.1500	3.5133	3.5988	40.7699
and $\alpha = 10$	(3)				−0.3495	−3.5574	−14.7800	0.4347	5.5990	51.9800
	(4)				−0.1010	−0.8020	−2.2370	0.1273	1.2032	10.2560
	(1)	0.0209	0.1286	0.4573	0.0134	0.0829	0.2994	0.0301	0.1976	0.6486
$m = 1$	(2)	−0.0209	−0.1286	−0.4573	−0.0134	−0.0829	−0.2994	−0.0301	−0.1843	−0.6486
and $\alpha = 0$	(3)				−35.9180	−35.5385	−34.5370	43.8990	53.6928	41.8260
	(4)				−35.9220	−35.5485	−34.5370	43.8990	43.3250	41.8260
	(1)	5.0844	5.5329	7.0335	5.0472	5.0021	12.3234	5.1325	7.1216	35.8050
$m = 1$	(2)	−5.0844	−5.5329	−7.0335	−5.0472	−5.0021	−12.3234	−5.1325	−7.1216	−35.8050
and $\alpha = 10$	(3)				−0.7330	−9.5950	75.2100	0.9450	28.7123	409.0600
	(4)				0.7330	−9.5900	75.2100	0.9450	28.7120	409.0600
	(1)	1.0434	1.2701	1.9951	1.0267	1.0467	1.4277	1.0641	1.6876	7.7082
$m = 2$	(2)	0.9488	0.6788	−0.1741	0.9688	0.9676	−0.1160	0.9226	0.7676	−9.1677
and $\alpha = 0$	(3)				−6.6010	−17.5900	−28.7400	1.9890	32.8700	286.3570
	(4)				2.1630	42.5500	−36.4590	−2.7060	13.0790	5166.3700
	(1)	6.1238	6.8042	9.1781	6.0634	6.0767	31.0285	6.2093	10.2321	76.0787
$m = 2$	(2)	−4.1391	−4.8871	−7.4726	−4.0687	−4.1138	−35.4462	−4.2341	−8.9198	−91.0675
and $\alpha = 10$	(3)				−1.0173	−10.6900	238.0690	1.3634	50.3792	161.8220
	(4)				−1.7010	−15.8225	374.3500	2.2950	82.5170	1118.6900

Table 2. Values of Nusselt numbers at the walls $y = 0$ and $y = 1$ and their percentage differences for QDT case

	γ	$\gamma = 0$ (LDT case)		$\gamma = 1$ (QDT case)	
		K			
m and α	$\overline{Nu}$	0.5	3	0.5	3
	$\overline{Nu}_0 = (1)$	0.9993	0.9958	0.9962	0.8732
$m = -1$	$\overline{Nu}_1 = (2)$	1.0007	1.0042	1.0038	1.1268
and $\alpha = 0$	$\overline{Nu}_{0DP} = (3)$			−0.3077	−12.3118
	$\overline{Nu}_{1DP} = (4)$			0.3068	12.2130
	(1)	−1.4138	−1.5849	−1.5160	−1.9045
$m = -1$	(2)	3.5088	3.5548	3.5088	3.6782
and $\alpha = 10$	(3)			0.1466	20.1683
	(4)			−28.4185	3.4796
	(1)	0.0209	0.1286	0.0208	0.1250
$m = 1$	(2)	−0.0209	−0.1286	−0.0208	−0.1250
and $\alpha = 0$	(3)			−4.6820	−2.7971
	(4)			−0.4777	−2.7971
	(1)	5.0844	5.5329	5.1348	18.2219
$m = 1$	(2)	−5.0844	−5.5329	−5.1348	−18.2219
and $\alpha = 10$	(3)			0.9900	229.3364
	(4)			0.9900	229.3364
	(1)	1.0434	1.2701	1.0506	3.4258
$m = 2$	(2)	0.9488	0.6788	0.9356	−2.9152
and $\alpha = 0$	(3)			0.6904	169.7355
	(4)			−1.3371	−529.4561
	(1)	6.1238	6.8042	−2.1536	−134.2290
$m = 2$	(2)	−4.1391	−4.8871	−4.3073	−44.7430
and $\alpha = 10$	(3)			2.2674	484.2861
	(4)			4.0617	815.5264

significant result which encourages further investigation over the effect of the NDT-parameter, on the flow and heat transfer. We note further that an increase in the wall temperature ratio or the free convection parameter or the heat source parameter contributes significantly to the effect of the NDT parameter on the heat flow. Also when the free convection parameter is large enough and when $\alpha = 0$ or 10 the heat flow to the wall $y = 0$ in the presence of positive γ is about five times that observed in its absence (at the other wall, this effect is about eighty times). The foregoing observations hold in all the cases except when $m = -1$; that is, when the average of wall temperatures $[(T_{w_1} + T_{w_0})/2]$ equals the temperature T_s of the static fluid. In this case of exception also, the effect of NDT-parameter taking positive values, definitely not insignificant, is qualitatively opposite in effect to the other cases $m = 1$ and 2. It is noticeable from Table 1 that

Table 3. Dimensionless velocity and temperature for QDT case

K, m and α	γ / y	$\gamma = 0$ (LDT case) u_2	t_3	$\gamma = 1$ (QDT case) $\bar{u}_2$	$\bar{t}_3$
	0.0	0.0000	3.0000	0.0000	3.0000
	0.2	0.3436	3.6945	0.9648	4.5571
$K = 3, m = 2$	0.4	0.5399	4.3115	1.6488	5.3972
and $\alpha = 0$	0.6	0.5639	4.9109	1.8648	5.9826
	0.8	0.3917	5.4987	1.3968	6.4782
	1.0	0.0000	6.0000	0.0000	6.0000
	0.0	0.0000	3.0000	0.0000	3.0000
	0.2	0.5904	6.2670	3.6729	19.1080
$K = 3, m = 2$	0.4	0.9350	8.1140	6.1545	23.7214
and $\alpha = 10$	0.6	0.9590	8.7130	6.4944	24.1930
	0.8	0.6380	8.0730	4.3181	21.5550
	1.0	0.0000	6.0000	0.0000	6.0000

when γ is negative, the values of the Nusselt numbers are decreased considerably over their LDT counterparts. Finally we mention from Fig. 6 that for $m = 2$, $\alpha = 0$ and K sufficiently large, Nu_1 changes sign from positive to negative—a result qualitatively interesting in that the heat which was flowing hitherto from the fluid to this wall, can now flow from the wall to the fluid (see Sinha [4]).

QDT case

It is evident that from the boundary value problem (4)–(5) of the NDT-case, we can obtain as a special case the problem discussed by Sinha [4]—the QDT case, but with the heat sources taken into account; provided (i) we delete the t-term from, and replace γt^2 by $t^2(\gamma = 1)$, in the equation (4) and (ii) we replace β_0 by $\beta_1 \Delta T$ (ΔT = a suitable quantity having the dimension of temperature) in the non-dimensionalisation and redefine the free convection parameter K in a suitable way.

In this case we label the non-dimensional velocity as $\bar{u}$, the temperature as $\bar{t}$ and Nusselt numbers as $\overline{Nu}_{0,1}$ and obtain solutions for $\bar{u}$ and $\bar{t}$ as in the NDT case. It is observed that the fluid-velocities and temperatures of this case are higher than their LDT counterparts—a phenomenon qualitatively in agreement with the one observed by Sinha [4]. The quantitative difference is that this increase in fluid velocity or temperature is more pronounced when the heat sources are present than in their absence (see Table 3, $m = 2$)—a result similar to that in NDT case.

From Table 2 we observe that for $m = 2$, the Nusselt number $\overline{Nu}_0(\overline{Nu}_1)$ increases (decreases) over its LDT counterpart regardless whether the heat sources are present or not—a result, again, in agreement with that of Sinha [4]. But when $\alpha = 10$ this increase or decrease is considerably high (see Table 2, $m = 2$).

The values of $\overline{Nu}_0$ are positive almost always and those of $\overline{Nu}_1$ change their sign from positive to negative. Physically it means that the heat flows to the wall $y = 0$ and, at the other wall, there is a possibility for the heat to flow either way (see Table 2, $m = 2$). This behaviour gets reversed in the case when $m = -1$. Finally we conclude that in the QDT-case the heat flow at the wall $y = 0$ is always into it like in the NDT-case while at the other wall, the heat flow can be at times into the wall and at other times into the fluid (see Table 2, $m = 2$)—a result in contrast to that in the NDT-case when the heat sources are present. For possible experimental verification we present in Table 1 the percentage-deviations of the wall Nusselt numbers of the NDT case from their LDT counterparts.

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CONVECTION NATURELLE LAMINAIRE PLEINEMENT DEVELOPPEE ENTRE DEUX PAROIS PARALLELES ET VERTICALES—(1ERE PARTIE)

Résumé—On analyse la convection naturelle et le transfert thermique entre deux parois parallèles et verticales, à partir de la variation non linéaire de la masse spécifique en fonction de la température (*NDT*) dans le terme de pesanteur. Lorsque le rapport *m* des températures pariétales, ou le paramètre *k* de convection naturelle, est modérément élevé les vitesses et les températures du fluide dans le cas *NDT* excèdent leurs homologues dans le cas de la variation linéaire de la masse spécifique et cet acroissement est plus prononcé en présence de sources de chaleur. On discute en particulier le problème lié à une variation quadratique de la masse spécifique en fonction de la température (*QDT*), en présence de sources de chaleur.

VOLLAUSGEBILDETE LAMINARE, FREIE KONVEKTIONSSTRÖMUNG ZWISCHEN ZWEI PARALLELEN, VERTIKALEN WÄNDEN. TEIL I

Zusammenfassung—Basierend auf der nicht-linearen Temperaturabhängigkeit der Dichte (*NDT*) im Auftriebsterm wird die freie Konvektionsströmung und der Wärmeübergang zwischen zwei parallelen, vertikalen Wänden untersucht. Bei großen Werten des Wandtemperaturverhältnisses *m* oder des Parameters *k* der freien Konvektion übersteigen die Fluidgeschwindigkeiten und die Temperaturen im *NDT*-Fall diejenigen, welche sich bei linearer Temperaturabhängigkeit der Dichte (*LDT*) ergeben. Sind Wärmequellen vorhanden, dann ist dieser Unterschied stärker ausgeprägt. Als Spezialfall wird das Problem mit einer quadratischen Temperaturabhängigkeit der Dichte (*QDT*) und Wärmequellen untersucht.

ПОЛНОСТЬЮ РАЗВИТОЕ ЛАМИНАРНОЕ СВОБОДНОКОНВЕКТИВНОЕ ТЕЧЕНИЕ МЕЖДУ ДВУМЯ ПАРАЛЛЕЛЬНЫМИ ВЕРТИКАЛЬНЫМИ СТЕНКАМИ. ЧАСТЬ I.

Аннотация — Для случая нелинейного изменения плотности в зависимости от температуры анализируется свободноконвективное течение и теплообмен между двумя параллельными вертикальными стенками. При относительно больших значениях отношения температур стенок *m* или параметра свободной конвекции *k* скорость жидкости и температура при нелинейном изменении плотности в зависимости от температуры выше их соответствующих значений при линейном изменении плотности, что особенно проявляется при наличии источников тепла. Как частный случай рассматривается задача о квадратичной зависимости плотности от температуры при наличии источников тепла.